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Conceptualization of Numbers, Number Operations, and Algebra among 10th and 11th Grade Students

Abstract *The current study explores the conceptual understanding of secondary level students in mathematics. The participants were 1320 students of 10th grade and 180 students of 11th grade randomly selected from 33 public secondary and 17 higher secondary schools of Lahore. The article describes the performance of students on six grade 1-7 level questions related to numbers, number operations, and algebra. One question was knowledge level and rest five were analysis level questions. Four questions were related to Knowledge of primary level Content and Students that required participants to analyze the error or explanations related to the problem. The other two questions were related to middle school content knowledge. The questions were adapted from Hill et. al. (2004)'s instrument of Study of Instructional Improvement (SII). The success rate of students was ranging from 52.8 to 63.8 % on six questions and from 22 to 73.5 % on 15 sub-questions.*

Key Words: Whole Numbers, Number Operations, Integers, Rational Numbers, Algebra, Error Analysis, Mathematical Thinking.

Introduction

Conceptual learning in mathematics focuses on ideas and generalizations that make connections among ideas. Procedural learning focuses on step by step procedures and skills (Ashlock, 2010). Conceptual knowledge involves explicit or implicit understanding of the principles and interrelations in a domain. Procedural knowledge, on the other hand, is knowing the steps or sequence of actions for solving problems. These two types of knowledge represent two distinct but interrelated ends of a continuum. Procedures used by children are based on their conceptual understanding (Rittle-Johnson & Alibali, 1999). Hence, the conceptual understanding of mathematics is very important.

Error analysis in mathematics is identifying patterns of misunderstanding in students' work to provide them instruction according to the area of their need. It is to pinpoint and correctly identify the specific errors and their causes when students make a consistent mistake. Errors in mathematics may occur due to carelessness, poor attention and lack of knowledge (Lai, 2012). Piaget suggests that errors made by the students during solving mathematical problems give important information related to their way of thinking (Ulu, 2017).

Errors made by the students may be slips or bugs. Computational errors are categorized under slips whereas misunderstanding under the bugs (Ashlock, 2010). Errors in mathematics can be factual, procedural, or conceptual. Factual errors are the mistakes when students cannot recall a fact required to solve a problem or do not have mastery of basic facts. Procedural error is when the correct procedure or steps are not followed for solving a problem. Slips include both procedural and factual errors. These are due to visual-motor integration problems, impulsivity, or memory deficits and are easier to identify. Inherent misunderstandings are not involved in these errors. Bugs or conceptual errors may look like procedural errors, but they are when a student does not have a complete understanding of a particular concept in mathematics, place value for instance (Ginsburg, 1987, as cited in Lai, 2012).

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Errors should be taken as opportunities rather than disadvantages as they reveal gaps in students' understandings. Students may develop understanding by bridging these gaps (Wischgoll, et al., 2015). Positive effects of the use of error analysis are found in students' learning (Ashlock, 2010; Izen, 1999; Martinez, 1998). Errors and the reasons for making these errors should be explored for identifying the progress of students (Ashlock, 2010) and making their learning more effective (Safriani et. al., 2019). The integration of errors in the learning process helps the learners in enhancing their problem-solving skills (Martinez, 1998). The use of errors promotes problem-solving through the use of creative thinking because errors motivate the students to identify what went wrong and act as catalysts to find new exciting ideas and a better approach. In this way, errors can positively change student thinking (Hetland, 2013). With this perspective of errors as a source of learning, students may develop an optimistic attitude towards learning (Eggleton & Moldavan, 2001; Martinez, 1998) and may learn mathematics without fear of making mistakes (Izen, 1999).

Traditionally, pedagogy of mathematics education relied on demonstrating correctly solved examples for students so that they can follow these as models while solving their mathematics problems (Atkinson et. al., 2000). A recent trend is to introduce incorrect exercises to students so that they can conduct an error analysis (McLaren et. al., 2015). Using both correctly solved exercises and incorrect exercises for error analysis leads to a greater understanding of mathematics (Rushton., 2018).

The Objective of the Study

The study aims at identifying 10^{th} and 11^{th}-grade students' conceptual understanding of whole numbers, number operations, integers, and algebra.

Research Methodology

Descriptive research was conducted to identify the conceptual understanding of secondary level students in mathematics. The population comprises 10^{th} and 11^{th}-grade students of public secondary and higher secondary schools of Lahore. A total of 1500 students randomly selected from 33 secondary and 17 higher secondary schools were the sample of the study. Among these, 1320 students belonged to grade 10 and 180 students belonged to grade 11. The study reports the analysis of six questions (15 sub-questions) adapted from items developed by Hill et al. (2004). The four questions were related to Knowledge of primary level Content and Students (KCS). KCS is knowing both students and mathematics and includes knowing and recognizing knowledge of common errors of students (Ball et al., 2008). Hence, three of these questions require error analysis. One question requires the analysis of explanations regarding a pattern. The last two questions were related to middle school content knowledge. The data were analyzed using descriptive statistics.

Results

Questions were analyzed in terms of grade-level expectations from the students as suggested by the Government of Pakistan (2006) and the cognitive level required for attempting these questions.

Table 1 gives a summary of the analysis of all the six questions in terms of curricular and cognitive levels required to attempt these questions.

Table 1. Content, and Curricular and Cognitive Levels of Questions

Question	Content and curricular level	Cognitive level
Place value	Grade 1 Concept of whole numbers: Identify the place value of a specific digit in a 2-digit number.	Analysis
Plus sign pattern	Grade 2 Number operations: Addition of two-digit numbers with carrying.	Analysis

Addition of columns of numbers	Grade 2 Number operations: Addition of two-digit numbers with carrying.	Analysis
Subtraction of columns of numbers	Grade 2-4 Number operations: Subtract ones from 3-digit numbers with borrowing (grade 2). Subtract numbers up to four digits with and without borrowing (grade 3). Subtract numbers up to 6-digit number (grade 4).	Analysis
Two negatives make positive	Grade 6 Integers: Addition, subtraction, multiplication, and division of integers with like and unlike signs	Knowledge
Distributive property of multiplication over addition	Grade 6-7 Whole number: Verify distributive property of multiplication over addition (grade 6). Operations on Rational Numbers Verify the distributive property of rational numbers concerning multiplication over addition/ subtraction (grade 7). Introduction to algebra: Combining of like terms (grade 6). Linear equations: Solve simple linear equation (grade 6).	Analysis

The correct responses are shown in bold in all tables.

Question1: Place value. Sanny was asked to show the number of checkers represented by the 3 in 23, he counted out 3 checkers. When he was asked to show the number of checkers represented by the 2 in 23, he counted out 2 checkers. What problem is Sanny having here:
Table 2 gives the options students provided with and percentage of students selecting these options.

Table 2. Options and Percentage of Students Selecting These Options

Sr No	Responses	Percentage
1	Sanny does not know how large 23 is.	2.2
2	Sanny considers 2 and 20 as the same.	27.5
3	Sanny has no understanding of the place values in the number 23.	**63.8**
4	No response	6.5

Table 2 shows that the majority of the students correctly identified that Sanny has no understanding of place values in the two-digit number. Other students either did not correctly identify the reason or did not respond to the question.

Question 2: Equality of pieces of plus sign pattern on the chart of 100.

In the plus sign on the chart of 100, the sum of the numbers in both the horizontal and vertical lines are the same (31 + 32 + 33 = 22 + 32 + 42). Which explanations show an adequate understanding of why this is true for all the plus signs on this chart?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Table 3 shows different explanations and the percentage of students selecting these explanations.

Table 3. Percentage Responses of Students on the Explanations Regarding the Equality of Sum of Numbers in the Pieces of Plus Signs on the Chart of 100

Sr. No	Explanations	Yes	No	Not sure	No Response
1	The average of the three horizontal and three vertical numbers is equal	**73.5**	15.5	06	5
2	The Sum of both vertical and horizontal lines is 96.	68.5	**22**	05	4.5
3	The sum of either piece of the plus sign is equal to three times of the middle number in all the plus signs	**54.4**	35.3	1.8	8.5
4	The vertical numbers are 10 more and 10 less than the middle number.	**68.5**	25.6	2.1	3.8
	Mean	**66.22**	24.6	3.7	4.45

Students were required to analyze the explanations of <u>why</u> the equality of the sum of horizontal and vertical pieces is true for all the plus signs on the chart of 100. For each plus sign on the chart of 100, set of numbers read vertically, 10 is subtracted from and 10 is added to the central number. Thus, $X - 10 + X + X + 10 = 3X$, the sum read must be three times the central number. For each plus sign on this chart, the numbers read horizontally, 1 is subtracted from and 1 is added to the central number. So, whenever read horizontally, it would be $X - 1 + X + X + 1 = 3X$. So, the average of the three horizontal and three vertical numbers would be equal in all the plus signs. Also, either piece of the plus sign would be equal to three times the middle number in all the plus signs. Explanation 2 is correct only for the highlighted plus sign but not for all the plus signs on this chart.

Question 3: Identifying errors in adding columns of numbers.

In the following three cases of the addition of columns of numbers

$$\text{I)}\ \begin{array}{r} ^{1}\\ 38\\ 49\\ \underline{+\,65}\\ 142 \end{array} \qquad \text{II)}\ \begin{array}{r} ^{1}\\ 45\\ 37\\ \underline{+\,29}\\ 101 \end{array} \qquad \text{III)}\ \begin{array}{r} ^{1}\\ 32\\ 14\\ \underline{+\,19}\\ 64 \end{array}$$

Which cases have a similar error?

Table 4 gives the options students were provided with and the percentage of students selecting these options.

Table 4. Identifying Errors in Adding Columns of Numbers

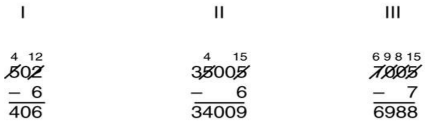
Sr. No.	Cases	Percentage

1	1 and 2	**63.4%**
2	1 and 3	1.6%
3	2 and 3	8.6%
4	1, 2 and 3	18.7%

Students in all three cases have the concept of place value. In case 3, there was only an error in mental arithmetic. The correct procedure is followed but the one's column is added incorrectly. In cases 1 and 2 of this question, one's column is added correctly but one ten is carried instead of two tens. The error is procedural in cases 1 and 2 and lays in knowing how many 10s to carry.

Question 4: Identifying errors in subtracting columns of numbers.

In the Following are three student mistakes in subtracting columns of numbers:



Which cases have a similar error?

Table 5 gives the options that students were provided with and the percentage of students selecting these options.

Table 5. Identifying Errors in Subtracting Columns of Numbers

Sr. No.	Cases	Percentage
1	1 and 2	**53.3%**
2	1 and 3	10.5%
3	2 and 3	21.4%
4	1, 2 and 3	12.9%

In all the options of this question, one ten was borrowed where the value of ten of the minuends was 0. The probability that students should choose case 3 as one of the pair to be chosen is almost even (46.7 to 53.3). The question, though is why was the case 3 different from the other two? In case 3, one ten was borrowed but two tens were subtracted at tens column of the minuend where the value of tens of the minuend was zero. When the case 3 gets to the tens and hundreds, he correctly sees that he has to borrow a thousand, reducing the thousands by the '1' he has borrowed, giving 9 in the column of hundreds. So, this shows that case 3 does have the concept of place value. The only mistake was in the tens column where he recorded '8' instead of '9'. This suggests that the error in case 3 is only procedural.

One ten was borrowed and incorrectly subtracted from hundreds in case 1 due to the presence of 0 at tens place and from thousands in case 2 due to the presence of 0 at places of tens and hundreds. Both cases 1 and 2 know that they have to borrow something to make up for the fact that 2 and 5 are less than 6. They both leave a record (12 and 15) of the first correct subtraction they made. But then they both fail to realize which of the tens, hundreds, and in case 2 the thousands, they should go to, to complete the subtraction process. They just run along until they find a number they can subtract '1' from. Hence, they both make a conceptual error as they cannot handle in their heads the fact of place values of each of the numbers. Both of them do not have a concept due to which they are making an error.

Question 5: Knowledge regarding the Application of rule "two negatives make a positive".

Students sometimes partly remember a rule. This question assesses if students know the operations for which this rule is and is not applicable.

Table 6 gives the percentages of students selecting various responses.

Table 6. Students Responses Regarding the Application of the Rule "Two Negatives Make a Positive" for the Four Basic Number Operations

Sr. No.	Operations	Sometimes true	Always true	Never true	Not sure	No response
1	Addition	7.5	21.3	**48.2**	12.3	10.7
2	Subtraction	7.6	21.6	**47.9**	12.1	10.8
3	Multiplication	5.9	**66**	18.6	3.8	5.7
4	Division	10.1	**57.7**	20.9	4.8	6.5

This is a grade 6 level and knowledge level question. unfortunately, many 10th and 11th-grade students have misconceptions regarding the applicability or non-applicability of the rule on four basic operations especially in case of addition and subtraction as evident from Table 6.

Question 6.

This question requires students to identify the situations where the distributive property of multiplication over addition [a (b + c) = ab + ac] is applicable or not applicable.
Table 7 gives various situations regarding the applicability or non-applicability of the distributive property of multiplication over addition and percentage responses of students on these situations.

Table 7. Identification of Situations for the Demonstration of Distributive Property of Multiplication over Addition

Sr No	Situations	Applicable	Not applicable	Not Sure	No Response
1	Adding 3/4 + 5/4	**55.9**	25.7	8.2	10.2
2	Solving x in 2x -5 = 8	16.8	**62.4**	11.6	9.2
3	$3x^2$+ 4y+$2x^2$- 6y combining like terms	46.7	**37.1**	7.3	8.9
4	Adding 34+25 in this way 34 +<u>25</u> 59	17.7	**55.8**	14.9	11.6

For item 1, students have to recognize that they have to decompose the expression into ¼ (3 + 5) and hence the property is applicable. Solving 2x - 5 = 8 for x (2x = 8 + 5 = 13, x = 13/2) does not require the application of this property and students found this item to be the easiest. The property may apply for combining the algebraic expressions but does not apply on combining the like terms in the expression given in item3: $3x^2$+ 4y + $2x^2$ - 6y = $3x^2$ + $2x^2$ + 4y -6y = $5x^2$-2y. It fits the x^2 part of the question: x^2 (3 + 2) but does not fit the entire question. However, this item was proved to be the hardest for the students. Item 4 is the addition of simple 2-digit numbers but surprisingly, a little less than half of the students could not identify that this does not require the application of this property.
Table 8 summarizes percentage correct responses of students on various items and subitems.

Table 8. Correct Percentage Responses of Students on Items and Subitems

<u>Question</u>	<u>M</u> <u>Subitem</u>	<u>M</u> <u>Item</u>
1. Place value		63.8
2. Plus sign pattern	1)73.5 2) 22 3) 54.4	

	4) 68.5	54.6
3. Addition of columns of numbers		63.4
4. Subtraction of columns of numbers		53.3
5. Two negatives make positive	1)48.2	
	2) 47.9	
	3) 66	
	4) 57.7	54.95
6. Distributive property of multiplication over addition	1) 55.9	
	2) 62.4	52.8
	3) 37.1	
	4) 55.8	

Table 8 shows that the success rate of students was ranging from 52.8 to 63.8 % on various items and from 22 to 73.5 % on subitem. The results showed that secondary school students performed better in identifying the error related to place value and the error in adding columns of numbers. Students' performance was comparatively poor in the rest of the questions.

Discussion

The current study analyses the conceptual understanding of 10th and 11th-grade students in mathematics. The six questions given to the students were related to whole numbers, number operations, integers, and algebra. Four questions were related to knowledge of primary level content and students. Students were asked to identify errors related to place value, to analyze the statements related to plus sign pattern on the chart of 100, and to identify an error besides and subtraction of columns of numbers. The first three questions were grade 1-2 level questions and question 4 was grade 2-4 level questions. The one middle school content knowledge question was grade 6 level and the other was grade 6-7 level question.

Students' performance was better in questions 1 and 3, that was related to the error analysis regarding place value and addition of columns of numbers. In the rest of the four questions, a little more than half students were able to perform correctly. Question 5 was a knowledge level question and surprisingly, less than half of the 10th and 11th-grade students could identify that the "two negatives make a positive" rule is not for addition and subtraction. However, comparatively a better proportion of students could identify that this rule is for multiplication and division. Question 6 required the students to identify the situations suitable for the demonstration of the distributive property of multiplication over addition. Students found it easiest that this property is not applicable for solving the equation for x and found it hardest this property is not applicable for combining the like terms in the given expression.

Conclusion

In this study, researchers explored 10th and 11th-grade students' conceptual understanding of whole numbers, number operations, integers, and algebra. Students were given six questions to attempt. The four questions are related to primary content and student knowledge. Questions 1-3 are grade 1-2 level questions and question 4 is a grade 2-4 level question. These questions required the students to analyze the error regarding the place value, the explanations regarding a pattern, and errors besides and subtraction of columns of numbers. The one content knowledge question of middle school was a grade 6 level question and the other was grade 6-7 level question. They required the students to identify the applicability or non-applicability of a rule to operations and a property to situations.

The results showed that 10th and 11th-grade students performed better in identifying errors related to place value and addition of columns of numbers. Students' performance was comparatively poor in the rest of the four items and a little more than half of students were able to correctly perform on these.

Many students were not able to identify that “two negatives make a positive” rule is for multiplication and division and not for addition and subtraction. Students found it easiest that the distributive property of multiplication over addition is not applicable for solving the equation for x and found it hardest that the property is not applicable for combining the like terms in the given expression.

Results suggest that students must be given opportunities for mathematical thinking. They should be given opportunities to explore strategies used to solve mathematical problems in the correct and incorrect exercises so that they have a sound understanding of mathematical concepts. Students will likely learn the more complex concepts in mathematics effectively when they have a good understanding of basic mathematical concepts.

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